Dynamical linked cluster expansions with applications to disordered systems

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Abstract. Dynamical linked cluster expansions are linked cluster expansions with hopping parameter terms endowed with their own dynamics. We discuss physical applications to systems with annealed and quenched disorder. Examples are the bond-diluted Ising model and the Sherrington-Kirkpatrick spin glass. We derive the rules and identify the full set of graphs that contribute to the series in the quenched case. This way it becomes possible to avoid the vague extrapolation from positive integer n to $n = 0$, that usually goes along with an application of the replica trick.

PACS. 75.10.Nr Spin-glass and other random models – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 02.10.-v Logic, set theory, and algebra

1 Introduction

Linked cluster expansions (LCEs) have a long tradition in statistical physics. Originally applied to classical fluids, later to magnetic systems ([1–3] and references therein), they were generalized to applications in particle physics in the eighties [4]. There they have been used to study the continuum limit of a lattice Φ^4 field theory in four dimensions at zero temperature. In [5,6] they were further generalized to field theories at finite temperature, simultaneously the highest order in the expansion parameter was increased to eighteen. Usually the analytic expansions are obtained as graphical expansions. Because of the progress in computer facilities and the development of efficient algorithms for generating the graphs, it is nowadays possible to handle of the order of billions of graphs. The whole range from high temperatures down to the critical region becomes available, and thermodynamic quantities like critical indices and critical temperatures are determined with high precision (the precision is comparable or even better than in corresponding high quality Monte Carlo results) [6–9]. An extension of LCEs to a finite volume in combination with a high order in the expansion parameter turned out to be a particularly powerful tool for investigating the phase structure of systems with first and second order transitions by means of a finite size scaling analysis [10].

Linked cluster expansions are series expansions of the free energy and connected correlation functions about an ultralocal, decoupled theory in terms of a hopping parameter K. The corresponding graphical representation is a sum in terms of connected graphs. The value of K parametrizes the strength of interactions between fields at different lattice sites. Usually they are chosen as nearest neighbours. In contrast to the ultralocal terms of a generic interaction we will sometimes refer to hopping terms as non-ultralocal.

In [14] we have generalized these linked cluster expansions to dynamical linked cluster expansions (DLCEs). These are linked cluster expansions with hopping parameter terms that are endowed with their own dynamics. Such systems are realized in spin glasses with (fast) spins and (slow) interactions [11–13]. They also occur in variational estimates for SU(N)-gauge-Higgs systems, *cf.* [14]. Like LCEs they are expected to converge for a large class of interactions.

Formally DLCEs amount to a generalization of an expansion scheme from 2-point to point-link-point-interactions. These are interactions between fields associated with two points and with one pair of points called link. The points and links are not necessarily embedded on a lattice, and the links need not be restricted to nearest neighbours. In [14] we have developed a new multiple-line graph theory in which a generalized notion of connectivity plays a central role. Standard notions of equivalence classes of graphs like 1-line irreducible and 1-vertex irreducible graphs have been generalized, and new notions like 1-multiple-line irreducible graphs were defined in order to give a systematic classification.

In this paper we discuss applications to disordered systems, in particular we show explicitly how to avoid the replica trick in quenched disordered systems. The paper is organized as follows. In Section 2 we specify the models that admit a DLCE. We introduce multiple-line graphs and explain the idea behind the abstract notions of multiple-line graph theory. Applications to disordered systems are presented in Section 3. There it is of particular interest that DLCEs allow for the possibility of avoiding the replica trick. In the quenched limit we derive that DLCEs must be restricted to a subclass of the corresponding graphical expansion, so-called quenched DLCEs (QDLCEs). We also list some examples for models whose phase structure is accessible to QDLCEs. In particular the bond diluted Ising model belongs to these examples.

2 A primer to DLCEs

To make the paper self-contained we first specify the class of models for which we have developed dynamical linked cluster expansions in [14]. Next we illustrate some basic notions of multiple-line graph theory, in particular the need for a new notion of connectivity.

By Λ_0 we denote a finite or infinite set of points. One of its realizations is a hypercubic lattice in D dimensions, infinite or finite in some directions with the topology of a torus. Λ_1 denotes the set of unordered pairs (x, y) of sites $x, y \in \Lambda_0$, $x \neq y$, also called unoriented links, and $\overline{\Lambda}_1$ a subset of Λ_1.

We consider physical systems with a partition function of the generic form

$$
\begin{aligned}
Z(H, I, v) &\equiv \exp W(H, I, v) \\
&= \mathcal{N} \int \mathcal{D}\phi \mathcal{D}U \exp\left(-S(\phi, U, v)\right) \\
&\quad \times \exp\left(\sum_{x \in \Lambda_0} H(x)\phi(x) + \sum_{l \in \overline{\Lambda}_1} I(l)U(l)\right),
\end{aligned}
$$
(1)

with measures

$$
\mathcal{D}\phi = \prod_{x \in \Lambda_0} \mathrm{d}\phi(x) \quad , \quad \mathcal{D}U = \prod_{l \in \overline{\Lambda}_1} \mathrm{d}U(l)
$$
(2)

and action

$$
\begin{aligned}
S(\phi, U, v) &= \sum_{x \in \Lambda_0} S^0(\phi(x)) + \sum_{l \in \overline{\Lambda}_1} S^1(U(l)) \\
&\quad - \frac{1}{2} \sum_{x, y \in \Lambda_0} v(x, y)\phi(x)U(x, y)\phi(y),
\end{aligned}
$$
(3)

with non-ultralocal couplings

$$
\begin{aligned}
v(x, y) &= v(y, x) \neq 0 \qquad \text{only for } (x, y) \in \overline{\Lambda}_1, \\
&\text{in particular } v(x, x) = 0.
\end{aligned}
$$
(4)

For later convenience the normalization *via* $\mathcal{N}$ is chosen such that $W[0, 0, 0] = 0$.

The field $\phi(x)$ is associated with the sites $x \in \Lambda_0$ and the field $U(l)$ lives on the links $l \in \overline{\Lambda}_1$, and we write $U(x, y) = U(l)$ for $l = (x, y)$. For definiteness and for simplicity of the notation here we assume $\phi(x) \in \mathbf{R}$ and $U(l) \in \mathbf{R}$. In our actual applications to spin glasses the ϕs are the (fast) Ising spins and the Us $\in \mathbf{R}$ are the (slow) interactions. The action is split into two ultralocal parts, S^0 depending on fields on single sites, and S^1 depending on fields on single links $l \in \overline{\Lambda}_1$. For simplicity we choose S^1 as the same function for all links $l \in \overline{\Lambda}_1$. We may identify $\overline{\Lambda}_1$ with the support of v,

$$
\overline{\Lambda}_1 = \{l = (x, y) | v(x, y) \neq 0\} \cdot
$$
(5)

The support of $v(x, y)$ need not be restricted to nearest neighbours, also the precise form of S^0 and S^1 does not matter for the generic description of DLCEs, S^0 and S^1 can be any polynomials in ϕ and U, respectively. The only restriction is the existence of the partition function.

Note that the interaction term $v(x, y) \ \phi(x) \ U(x, y) \ \phi(y)$ contains a point-link-point-interaction and generalizes the 2-point-interactions $v(x, y) \ \phi(x) \ \phi(y)$ of usual hopping parameter expansions. The effective coupling of the ϕ fields has its own dynamics governed by $S^1(U)$, the reason why we have called the new expansion scheme *dynamical* LCE.

Dynamical linked cluster expansions are induced by a Taylor expansion of $W(H, I, v) = \ln Z(H, I, v)$ about $v = 0$, the limit of a completely decoupled system. We want to express the series for W in terms of connected graphs. Let us consider the generating equation

$$
\begin{aligned}
\partial W / \partial v(xy) &= 1/2\langle\phi(x)U(x, y)\phi(y)\rangle \\
&= \frac{1}{2}\bigg(W_{H(x)I(x,y)H(y)} + W_{H(x)H(y)}W_{I(x,y)} \\
&\quad + W_{H(x)I(x,y)}W_{H(y)} + W_{I(x,y)H(y)}W_{H(x)} \\
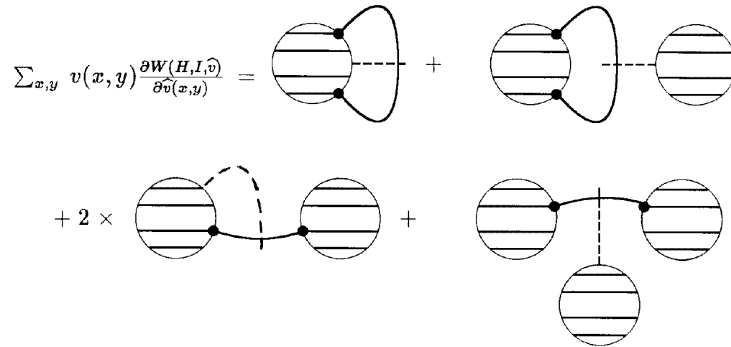&\quad + W_{H(x)}W_{H(y)}W_{I(x,y)} \bigg).
\end{aligned}
$$
(6)

Here $\langle\cdot\rangle$ denotes the normalized expectation value w.r.t. the partition function of equation (1). Subscripts $H(x)$ and $I(x, y) = I(y, x) = I(l)$ denote the derivatives of W w.r.t. $H(x)$ and $I(x, y)$, respectively.

Next we would like to represent the right hand side of equation (6) in terms of connected graphs. Once we have such a representation for the first derivative of W w.r.t. v, graphical expansions for the higher derivatives can be traced back to the first one.

For each W in equation (6) we draw a shaded bubble, for each derivative w.r.t. H a solid line, called a ϕ-line, with endpoint vertex x, and for each derivative w.r.t. I a dashed line, called a U-line, with link label $l = (x, y)$. The main graphical constituents are shown in Figure 1. Two ϕ-lines with endpoints x and y are then joined by means of a dashed U-line with label l, if the link l has x and y as its endpoints, *i.e.* $l = (x, y)$. According to these
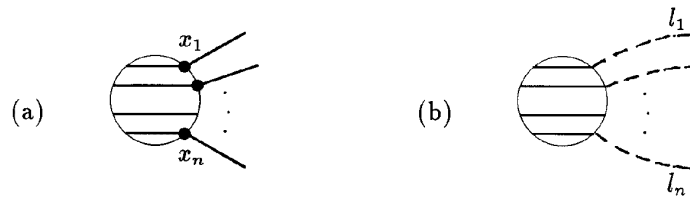
Fig. 1. Graphical representation of the derivatives of $W(H, I, v)$. (a) n-point function $\partial^n W/\partial H(x_1) \cdots \partial H(x_n)$, (b) n-link function $\partial^n W/\partial I(l_1) \cdots \partial I(l_n)$.

Fig. 2. Generating equation of the graphical expansion of DLCEs. The solid line in each graph carries a propagator $v(x, y)$. A dashed U-line with label l intersects a solid line with endpoints x and y if $l = (x, y)$.
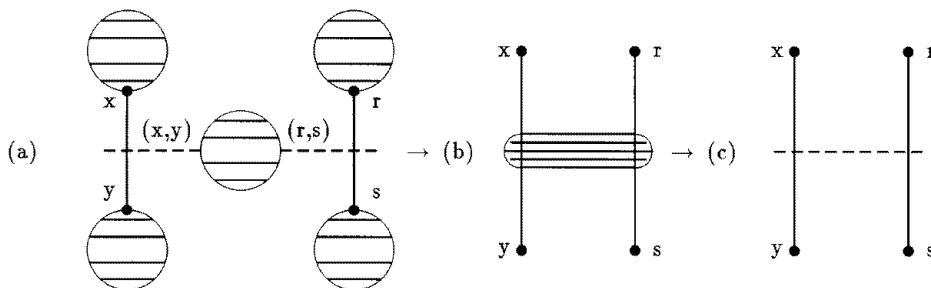
Fig. 3. Representation of $W_{H(x)}W_{H(y)}W_{H(r)}W_{H(s)}W_{I(x,y)I(r,s)}$. (a) according to the rules of Figures 1 and 2, (b) same as (a), but at $v = 0$ and simplified for a formal definition of multiple-line connectivity, *cf.* [14], (c) same as (b), but for use in the actual graphical representations.

rules equation (6), multiplied by $v(x, y)$ and summed over x and y, is represented by Figure 2. Note that, because of the Taylor operation, each solid line from x to y carries a factor $v(x, y)$.

Since the actual need for a new type of connectivity is not quite obvious from Figure 2, because equation (6) does not contain higher than first order derivatives w.r.t. I, let us consider a term

$$W_{H(x)}W_{H(y)}W_{H(r)}W_{H(s)}W_{I(x,y)I(r,s)} \qquad (7)$$

occurring in the second derivative of W w.r.t. $v(x, y)$, $v(r, s)$. According to the above rules this term would be represented as shown in Figure 3a. While the 2 vertices in the last term of Figure 2 are connected in the usual

sense *via* a common (solid) line (the dashed line with an attached bubble could be omitted in this case), the graph in Figure 3a would be disconnected in the old sense, since neither x nor y are line-connected with r and s, but – as a new feature of DLCE graphs – the lines from x to y and from r to s are connected *via* the dashed lines emerging from a common bubble shown in the middle of the graph. As we see from Figure 3a, we need an additional notion of connectivity referring to the possibility of multiple-line connectivity. While the analytic expression is fixed, it is a matter of convenience to further simplify the graphical notation of Figure 3a at $v = 0$. Two possibilities are shown in Figures 3b and c. To Figure 3b we later refer in the formal definition of the new type of multiple-line connectivity. In the familiar standard notion of connectivity two

vertices of a graph are connected *via* lines. The vertices are line-connected. Already there, in a dual language, one could call two lines connected *via* vertices. The second formulation is just appropriate for our need to define when two lines are connected. The corresponding vertices mediating the connectivity of lines are visualized by tubes, in Figure 3b we have just one of them. The tubes should be distinguished from the former type of vertices represented as full dots which are connected *via* bare ϕ-lines. In Figure 3c we show a simplified representation of Figure 3b that we actually use in graphical expansions.

The derivative terms have to be evaluated at $v = 0$. For $v = 0$ we have a decomposition of W according to

$$W(H, I, v = 0) = \sum_{x \in \Lambda_0} W^0(H(x)) + \sum_{l \in \overline{\Lambda}_1} W^1(I(l)) \quad (8)$$

with

$$\exp W^0(H) \equiv Z^0(H) = \frac{\int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-S^0(\phi) + H\phi\right)}{\int_{-\infty}^{\infty} \mathrm{d}\phi \exp\left(-S^0(\phi)\right)} \tag{9}$$

and

$$\exp W^1(l) \equiv Z^1(I) = \frac{\int_{-\infty}^{\infty} \mathrm{d}U \exp\left(-S^1(U) + IU\right)}{\int_{-\infty}^{\infty} \mathrm{d}U \exp\left(-S^1(U)\right)} . \tag{10}$$

In equations (9, 10) we have omitted any single site or single link dependence, because we assume that S^0 and S^1 are the same for all $x \in \Lambda_0$ and $l \in \overline{\Lambda}_1$, respectively. Therefore, at $v = 0$, the only non-vanishing derivatives of W are

$$W_{H(x_1)H(x_2)...H(x_n)}\big|_{v=0} = \frac{\partial^n W^0(H(x_1))}{\partial H(x_1)^n} \delta_{x_1, x_2, ... x_n} \tag{11}$$

and

$$W_{I(l_1)I(l_2)...I(l_m)}\big|_{v=0} = \frac{\partial^m W^1(I(l_1))}{\partial I(l_1)^m} \delta_{l_1, l_2, ... l_m}, \tag{12}$$

but mixed derivatives w.r.t. H and I vanish. As anticipated in Figures 3b and c, for $v = 0$ we replace the dashed bubbles and graphically distinguish between bubbles with ϕ-lines and bubbles with U-lines. We define

$$\left. \begin{array}{c} \end{array} \right\} n = v_n^{\circ c} = \left(\frac{\partial^n W^0(H)}{\partial H^n} \right)_{H=0} \tag{13}$$

for a connected n-point vertex with $n \geq 1$ bare ϕ-lines emerging from it and

$$\left. \begin{array}{c} \end{array} \right\} \nu = m_\nu^{1c} = \left(\frac{\partial^\nu W^1(I)}{\partial I^\nu} \right)_{I=0} \tag{14}$$

for a connected ν-line consisting of ν bare lines. If $\nu = 1$, we often omit the dashed line. If the bare lines of a ν-line are internal ϕ-lines, they get vertices attached to their endpoints, if they are external U-lines, no vertices will be attached.

Let V denote the lattice volume in D dimensions. The Taylor expansion of W about $v = 0$ to second order in v then reads

$$
\begin{aligned}
W(H, I, v) = {} & W(H, I, v = 0) \\
& + \sum_{x, y \in \Lambda_0} v(x, y) \frac{1}{2} W_{H(x)} W_{H(y)} W_{I(x,y)} \\
& + \frac{1}{2} \sum_{x, y, r, s \in \Lambda_0} \frac{1}{4} v(x, y) v(r, s) \\
& \times \Big(4\, W_{H(y)} W_{H(s)} W_{H(r)H(x)} W_{I(x,y)} W_{I(r,s)} \\
& + 2\, W_{H(x)H(r)} W_{H(y)H(s)} W_{I(x,y)} W_{I(r,s)} \\
& + 4\, W_{H(y)} W_{H(s)} W_{H(r)H(x)} W_{I(x,y)I(r,s)} \\
& + 2\, W_{H(r)H(x)} W_{H(y)H(s)} W_{I(r,y)I(x,s)} \\
& + W_{H(x)} W_{H(y)} W_{H(r)} W_{H(s)} W_{I(x,y)I(r,s)} \Big)_{v=0} \\
& + O(v^3),
\end{aligned} \tag{15}
$$

where we have used that $v(x, x) = 0$. For each W in the products of Ws we now insert equations (11, 12).

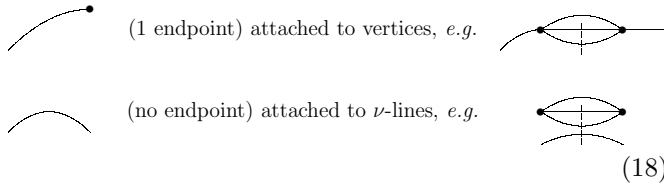If we choose v in a standard way as next-neighbour couplings

$$v(x, y) = 2K \sum_{\mu=0}^{D-1} (\delta_{x+\hat{\mu}, y} + \delta_{x-\hat{\mu}, y}) \tag{16}$$

with $\hat{\mu}$ denoting the unit vector in μ-direction, equation (15) becomes in a graphical representation at $H = I = 0$

$$+ O(K^3).$$

For clarity, here we have written explicitly the topological symmetry factors and the lattice embedding numbers. (Usually graphs represent their full weights including these factors.) Note that the first two graphs of the second order contribution also occur in a usual LCE with frozen U-dynamics, the next two differ by an additional dashed 2-line and the last one becomes even disconnected without the dashed line.

As usual, graphical expansions for correlation functions, in particular susceptibilities, are generated from $W(H, I, v)$ by taking derivatives w.r.t. the external fields H and I. Graphically this amounts to attaching external ϕ-lines and U-lines with

$$(18)$$

In passing we remark that the conventional LCE is included as a special case of the DLCE, if the U-dynamics is "frozen" to some value $U_0 \neq 0$, so that

$$
\begin{aligned}
W^1(I) &= -S_1(U_0) + IU_0, \\
\frac{\partial W^1(I)}{\partial I} &= U_0, \\
\frac{\partial^n W^1(I)}{\partial I^n} &= 0 \quad \text{for all } n > 1,
\end{aligned}
\tag{19}
$$

i.e., no n-lines do occur with $n > 1$. In this case it becomes redundant to attach dashed lines to bare lines. As mentioned above, in an LCE only the first three contributions would be left in equation (17).

3 Applications to disordered systems

In this section we consider applications of DLCEs to disordered systems, in particular to spin glasses with "slow" interactions coupled to "fast" spins. The interactions J are assumed to be in equilibrium with a thermal heat bath of inverse temperature β', while the spins σ are equilibrated according to a second inverse temperature β. Both systems need not be mutually in equilibrium. Let $Z_\beta(J)$ be the partition function that describes the equilibrium distribution of the spins for given Js,

$$
Z_\beta(J) = \sum_{\{\sigma_i = \pm 1\}} \exp\left(\beta \sum_{i<j} J_{(i,j)}\sigma_i\sigma_j\right).
\tag{20}
$$

The sum runs over pairs (i, j) that need not be restricted to nearest neighbours only.

We further assume that the dynamics of the time evolution of the slow interactions J is governed by a Langevin equation

$$
N\frac{\mathrm{d}}{\mathrm{d}t}J_{(i,j)} = -\frac{\partial}{\partial J_{(i,j)}}\mathcal{H}(J) + \sqrt{N}\eta_{ij}(t)
\tag{21}
$$

with

$$
\mathcal{H}(J) = -\frac{1}{\beta}\ln Z_\beta(J) + \frac{1}{2}\mu N \sum_{i<j=1}^N J_{(i,j)}^2.
\tag{22}
$$

Here $Z_\beta(J)$ is given by (20), N is the total number of spins, μ is a positive constant and η_{ij} is a stochastic Gaussian white noise of zero mean and correlation

$$
\langle \eta_{ij}(t)\eta_{kl}(t')\rangle = \frac{2}{\beta'}\delta_{(ij),(kl)}\delta(t - t').
\tag{23}
$$

Such a Langevin equation for the Js can be derived from an ansatz which is motivated by neural networks [11–13]. Moreover, since the time evolution of the Js is determined by a dissipative Langevin equation, the equilibrium distribution of the slow variables is again a Boltzmann distribution, governed now by the second temperature β'^{-1},

$$
Z'_{\beta'} = \mathcal{N}\int_{-\infty}^{\infty}\prod_{i<j=1}^N \mathrm{d}J_{(i,j)}\,\exp\left(-\beta'\mathcal{H}(J)\right),
\tag{24}
$$

with $\mathcal{N}$ some normalization that will be specified below. The effective Hamiltonian $\mathcal{H}$ of J is given by (22).

It is these equilibrium aspects of coupled systems of fast spins and slow interactions that we can treat analytically with DLCEs, as we will show below.

Let us first rewrite $Z'_{\beta'}$ in the form

$$
\begin{aligned}
Z'_{x\beta} &= \int_{-\infty}^{\infty}\prod_{i<j=1}^N \left(\sqrt{\frac{QN}{2\pi}}dJ_{(i,j)}\right) \\
&\quad \times \exp\left(-\frac{1}{2}QN\sum_{i<j}J_{(i,j)}^2\right)\,Z_\beta(J)^x \\
&\equiv [[Z_\beta(J)^x]],
\end{aligned}
\tag{25}
$$

where we have introduced $Q = \beta'\mu$ and real $x = \beta'/\beta$ as the ratio of two temperatures. The normalization has been chosen such that $[[1]] = 1$. If we assume that the time scale of fluctuations of the spins is so short that the J are only sensitive the averages of the σ, the quantity of physical interest is not

$$
\begin{aligned}
\ln\Bigg[&\int_{-\infty}^{\infty}\prod_{i<j=1}^N \left(\sqrt{\frac{QN}{2\pi}}dJ_{(i,j)}\right) \\
&\times \left(\sum_{\{\sigma_i\}}\exp\left(\beta \sum_{i<j=1}^N J_{(i,j)}\sigma_i\sigma_j\right)\right)^x \\
&\times \exp\left(-\frac{1}{2}QN\sum_{i<j}J_{(i,j)}^2\right)\Bigg]
\end{aligned}
\tag{26}
$$

where fluctuations of the σs do influence the Js, but

$$\int_{-\infty}^{\infty} \prod_{i<j=1}^{N} \left(\sqrt{\frac{QN}{2\pi}} dJ_{(i,j)} \right)$$

$$\times \ln \left[\sum_{\{\sigma_i\}} \exp \left(\beta \sum_{i<j=1}^{N} J_{(i,j)} \sigma_i \sigma_j \right) \right]$$

$$\times \exp \left(-\frac{1}{2} QN \sum_{i<j} J_{(i,j)}^2 \right), \quad (27)$$

or, in a shorthand notation,

$$\int \mathcal{D}J \ln Z_\beta(J) \equiv [[\ln Z_\beta(J)]]. \quad (28)$$

Usually one rewrites

$$\int \mathcal{D}J \ln Z_\beta(J) = \int \mathcal{D}J \lim_{x\to 0} \frac{Z_\beta(J)^x - 1}{x}$$

$$= \lim_{x\to 0} \int \mathcal{D}J \frac{Z_\beta(J)^x - 1}{x}$$

$$= \lim_{x\to 0} \frac{\ln\{1 + ([[Z_\beta(J)^x]] - 1)\}}{x}$$

$$= \lim_{x\to 0} \frac{\ln Z'_{x\beta}}{x}. \quad (29)$$

For the second equality sign one has assumed that $\int \mathcal{D}J$ commutes with $\lim_{x\to 0}$, while $\mathcal{D}J$ stays finite as $\beta' \to 0$. (Note that $\mathcal{D}J$ stays finite if $\mu \to \infty$ such that $Q = \beta'\mu$ stays finite for $\beta' \to 0$.) In the third equality sign it was assumed that $\lim_{x\to 0}[[Z_\beta(J)^x]] = 1$. Because of the equality (29) the limit $x \to 0$ is called the quenched limit. Stated in more physical terms, the spins look frozen from the point of view of the J, their dynamics does not influence the J, but *vice versa* the J-dynamics does influence the spins.

So far, x as the ratio of two temperatures is real. Rewriting the left hand side of (29) according to the right hand side is called the replica trick [17]. The uncontrolled approximation that usually enters the replica trick is that now the right hand side is evaluated for positive integer $x \equiv n$ and extrapolated to $n = 0$. Clearly a function that is known only for positive integer n does not have a unique extrapolation to $n = 0$ without further assumptions. Nevertheless, this approximation is made, because it is rather convenient. For integer n, $Z_\beta(J)^n$ is the partition function of an n times replicated system of which the logarithm is taken after the integration over the Js. It is seen as follows. We rewrite

$$Z_\beta(J)^n = \sum_{\{\sigma_i^{(a)}\}} \exp \left(\beta \sum_{a=1}^{n} \sum_{i<j=1}^{N} J_{(i,j)} \sigma_i^{(a)} \sigma_j^{(a)} \right), \quad (30)$$

with $a = 1, \ldots, n$ labelling the replicated spin variables, so that

$$Z'_{n\beta} = \int_{-\infty}^{\infty} \prod_{i<j=1}^{N} \mathrm{d}J_{(i,j)} \sum_{\{\sigma_i^{(a)}=\pm 1\}} \exp \left(-S(J, \sigma^{(a)}) \right),$$

$$S\left(J, \sigma^{(a)}\right) = -\beta \sum_{a=1}^{n} \sum_{i<j=1}^{N} J_{(i,j)} \sigma_i^{(a)} \sigma_j^{(a)} + \frac{1}{2} QN \sum_{i<j} J_{(i,j)}^2.$$

$$(31)$$

Linear terms in σ and J may be included according to

$$S_{lin} = -h \sum_{a=1}^{n} \sum_{i=1}^{N} \sigma_i^{(a)} + c \sum_{i<j=1}^{N} J_{(i,j)} \quad (32)$$

with constant external fields h and c.

Obviously, because of integer n, $Z'_{n\beta}$ has the form of models to which DLCE applies, with a hopping term

$$S_{hop}\left(J, \sigma^{(a)}\right) = -\beta \sum_{a=1}^{n} \sum_{i<j=1}^{N} J_{(i,j)} \sigma_i^{(a)} \sigma_j^{(a)}, \quad (33)$$

a single link action

$$S^1(J_{(i,j)}) = cJ_{(i,j)} + \frac{1}{2} QN \, J_{(i,j)}^2, \quad (34)$$

and a single site action

$$S^0\left(\sigma_i^{(a)}\right) = -h \sum_{a=1}^{n} \sigma_i^{(a)}. \quad (35)$$

Obviously the single link interaction (34) could be replaced by any suitably bounded function of the $J_{(ij)}$, in particular by non-quadratic monomials corresponding to non-Gaussian distributions, without rendering DLCEs impracticable.

Depending on n we distinguish the following cases.

- $n = 1$. First we note that for $n = 1$ we can directly apply DLCE to $\ln Z'_{\beta'=\beta}$ and to derived quantities to obtain their series expansions in β. But from a physical point of view, in a disordered system one is not interested in $n = 1$, because $n = 1$ corresponds to the completely annealed situation, in which the fast spins and the slow interactions are in mutual equilibrium. (In contrast, in particle physics one *is* interested in the $n = 1$ case, *cf.* our applications of DLCEs in the framework of variational cumulant expansions of the $SU(2)$ Higgs model [14].)

- $n > 1$, integer. Again we apply DLCE to $\ln Z'_{\beta'=n\beta}$, but have to account for the permutation symmetry between the replicas. Formally, the replica symmetry plays a role similar to an internal symmetry, *e.g.* an $O(N)$ symmetry in a scalar Higgs model. DLCEs with nontrivial internal symmetries have been discussed in connection with the $SU(2)$ Higgs model [14]. Thus we can study "unquenched" equilibrium

aspects of systems with two temperatures and compare the results from DLCEs adapted to "internal" replica symmetry with Monte Carlo simulations for the same n [18,19].

- $n = 0$, the quenched limit. As we will show in the next section, in order to discuss the $x \to 0$ limit, we need not refer to n times replicated systems $Z_\beta(J)^n$ characterized by (33–35), but just to $Z_\beta(J)$ given by (33–35) with $n = 1$. By means of special DLCEs, so-called quenched DLCEs (QDLCEs), we directly calculate the left hand side of (29). Therefore, setting $n = 1$ in (33–35) in QDLCEs does *not* imply the completely annealed case, because we first take the logarithm of $Z_\beta(J)$ and then average over the Js.

3.1 Avoiding the replica trick

In this section we give a detailed proof of the statement how to avoid the replica trick that has been made in [14]. First we adapt the notation to Section 2 to include more general cases. Λ_0 denotes the support of the spins, that is the set of lattice sites, with $V = |\Lambda_0|$ denoting their total number. $\overline{\Lambda}_1 \subseteq \Lambda_1$ are the pairs of sites whose spins interact. In accordance with (25), we write for the normalized link-average of a function $f(J)$

$$[[f(J)]] \ = \ \int \mathcal{D}J \ f(J) \tag{36}$$

with

$$\mathcal{D}J = \prod_{l \in \overline{\Lambda}_1} \mathrm{d}\mu(J(l)) \ ,$$

$$\mathrm{d}\mu(J) = \mathcal{N}_1 \ \mathrm{d}J \ \exp\left(-S^1(J)\right) \ , \ \int_{-\infty}^{\infty} \mathrm{d}\mu(J) = 1. \tag{37}$$

It is convenient to introduce the single link expectation values

$$\langle g(J) \rangle_1 \ \equiv \ \int \mathrm{d}\mu(J) \ g(J) \tag{38}$$

and the generating function $W^1(I)$ by

$$\exp W^1(I) \ \equiv \ \langle \exp(IJ) \rangle_1 \ . \tag{39}$$

The way in which the replica trick can be avoided is examplified for the free energy density W_{sp}/V of the spin system averaged over the link couplings. The partition function of the spin system for a given distribution of the link interactions $J(x,y)$ is given by

$$\exp W_{sp}(J) \ = \ \mathcal{N}_{sp} \int \mathcal{D}\sigma \ \exp\left(-S_{sp}(\sigma, J)\right), \tag{40}$$

where $W_{sp}(0) = 0$ and

$$S_{sp}(\sigma, J) = -\frac{1}{2} \sum_{x,y \in \Lambda_0} v(x,y)\sigma(x)\sigma(y)J(x,y),$$

$$\mathcal{D}\sigma = \prod_{x \in \Lambda_0} \mathrm{d}\sigma(x) \exp\left(-S^0(\sigma(x))\right). \tag{41}$$

Without loss of generality we identify the support of the interaction $v(x,y) = v(y,x)$ with the set $\overline{\Lambda}_1$ of lattice sites where $\mathcal{D}J$ is supported,

$$\overline{\Lambda}_1 \ = \ \left\{ l = (x,y) \in \overline{\Lambda_0 \times \Lambda_0} \mid v(x,y) \neq 0 \right\} \cdot \tag{42}$$

For simplicity we assume $v(x,y)$ to be of the form

$$v(x,y) \ = \ 2K \sum_{z \in \mathcal{N}(x)} \delta_{y,z}, \tag{43}$$

so that K is a measure of the strength of the interactions $v(x,y)$.

The free energy density of the spin system allows for a series expansion in the standard LCE sense, with the link field $J(l)$ playing the role of a "background field",

$$\frac{1}{V}W_{sp}(J) \ = \ \sum_{L \geq 0}(2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} w^{sp}(\Gamma, J). \tag{44}$$

Here $\mathcal{G}_E^{sp}(L)$ (with $E = 0$) denotes the set of equivalence classes of connected LCE graphs with E external lines and L internal lines. The spin-weights $w^{sp}(\Gamma, J)$ are of the form

$$w^{sp}(\Gamma, J) \ = \ R^{sp}(\Gamma) \sideset{}{'}\sum_{\mathcal{L}_\Gamma \to \overline{\Lambda}_1} \prod_{l \in \overline{\Lambda}_1} J(l)^{m(l)}. \tag{45}$$

The sum is taken over all non-vanishing lattice embeddings of the graph Γ. It runs over all maps of internal lines of the graph Γ to pairs of lattice sites of $\overline{\Lambda}_1$ that are consistent with the graph topology in the sense discussed in [14]. For every $l \in \overline{\Lambda}_1$, $m(l)$ denotes the number of lines of Γ that are mapped onto the link l by the embedding. All other factors that contribute to the weight are collected in the prefactor $R^{sp}(\Gamma)$, including the inverse topological symmetry number of Γ.

Next we want to express $[[W_{sp}(J)]]$ as a series in K by means of DLCE. Toward this end we set $f(J) = W_{sp}(J)$ and insert the series (44) with (45) into (36). At this stage we are not concerned with question of (uniform or dominated) convergence and obtain

$$\left[\left[\frac{1}{V} \ W_{sp}(J)\right]\right] = \sum_{L \geq 0}(2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} \int \mathcal{D}J \ w^{sp}(\Gamma, J)$$

$$= \sum_{L \geq 0}(2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} R^{sp}(\Gamma)$$

$$\times \sideset{}{'}\sum_{\mathcal{L}_\Gamma \to \overline{\Lambda}_1} \prod_{l \in \overline{\Lambda}_1} \left\langle J(l)^{m(l)} \right\rangle_1 \cdot \tag{46}$$

The next step is to express the full single link expectation values in terms of the connected ones. They are related by

$$\langle J^m \rangle_1 \ = \ \sum_{\Pi \in \mathcal{P}(\underline{m})} \prod_{P \in \Pi} \left\langle J^{|P|} \right\rangle_1^c, \tag{47}$$

where $\mathcal{P}(\underline{m})$ denotes the set of all partitions of $\underline{m} = \{1, \ldots, m\}$ into non-empty, mutually disjoint subsets of $\underline{m}$. $|P|$ is the number of elements of the set P. The relation (47) is equivalent to the partition of all lines of Γ that are mapped to the same lattice link into multiple lines, with every multiple line contributing a factor

$$\left\langle J^{|P|} \right\rangle_1^c = \left. \frac{\partial^{|P|} W^1(I)}{\partial I^{|P|}} \right|_{I=0} = m_{|P|}^{1c}. \qquad (48)$$

Using (47, 48) we rewrite (46) as

$$\left[\left[\frac{1}{V} W_{sp}(J)\right]\right] = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} R^{sp}(\Gamma)$$
$$\times \sum_{\Pi \in \mathcal{P}(\mathcal{L}_\Gamma)} \left(\prod_{P \in \Pi} m_{|P|}^{1c}\right) \left(\sideset{}{'}\sum_{\Pi \to \overline{\Lambda}_1} \prod_{l \in \overline{\Lambda}_1} 1\right). \qquad (49)$$

The last summation in (49) is over all maps $\mathcal{L}_\Gamma \to \overline{\Lambda}_1$ of the lines of Γ to the lattice links of $\overline{\Lambda}_1$ subject to the constraint that all lines that belong to the same multiple-line corresponding to some $P \in \Pi$ are mapped onto the same lattice link.

Finally we rewrite (49) as a sum over multiple-line graphs. To this end, we first observe that for every $\Gamma \in \mathcal{G}_0^{sp}(L)$, every partition $\Pi \in \mathcal{P}(\mathcal{L}_\Gamma)$ of the lines of Γ into multiple-lines generates a multiple-line graph $\Delta = (\Gamma, \Pi)$ in the obvious way. Let us denote by $\overline{\mathcal{G}}_{0,0}(L)$ the subset of multiple-line graphs of $\mathcal{G}_{0,0}(L)$ that stay connected after decomposition of all multiple lines. (These are the multiple-line graphs which stay connected in the usual graph theoretical sense, when the dashed lines are omitted.) For every $\Delta \in \overline{\mathcal{G}}_{0,0}(L)$ there is a unique $\Gamma(\Delta) \in \mathcal{G}_0^{sp}(L)$ and at least one $\Pi \in \mathcal{P}(\mathcal{L}_{\Gamma(\Delta)})$ such that $(\Gamma(\Delta), \Pi) = \Delta$. Let n_Δ be the (uniquely determined) number of partitions $\Pi \in \mathcal{P}(\mathcal{L}_{\Gamma(\Delta)})$ with $(\Gamma(\Delta), \Pi) = \Delta$, and $\Pi(\Delta)$ such an arbitrary partition. Equation (49) then becomes

$$\left[\left[\frac{1}{V} W_{sp}(J)\right]\right] = \sum_{L \geq 0} (2K)^L \sum_{\Delta \in \overline{\mathcal{G}}_{0,0}(L)} n_\Delta R^{sp}(\Gamma(\Delta))$$
$$\times \left(\prod_{P \in \Pi(\Delta)} m_{|P|}^{1c}\right) \left(\sideset{}{'}\sum_{\Pi(\Delta) \to \overline{\Lambda}_1} 1\right). \qquad (50)$$

The last bracket of (50) is the lattice embedding factor of the multiple-line graph Δ. The second bracket from the right does not depend on the choice of $\Pi(\Delta)$ and is the product of the multiple-line coupling constants as defined in [14]. Finally, $n_\Delta R^{sp}(\Gamma(\Delta))$ is precisely the remaining part of the weight of Δ that was described in detail in [14], endowed with the correct inverse topological symmetry number of the multiple-line graph Δ (because of the factor n_Δ).

In summary, we obtain the series expansion of the link-averaged free energy density in terms of DLCE graphs,

$$\left[\left[\frac{1}{V} W_{sp}(J)\right]\right] = \sum_{L \geq 0} (2K)^L \sum_{\Delta \in \overline{\mathcal{G}}_{0,0}(L)} w(\Delta). \qquad (51)$$

The weight $w(\Delta)$ of a multiple-line graph Δ is defined and computed according to the rules given in [14].

Equation (51) is the series representation of the link-averaged free energy density of the spin system, *i.e.* the free energy density of the $n = 0$ replica system, in terms of DLCE graphs. It looks much like the series representation of the 1-replica system, which is given by

$$\frac{1}{V} W_{1-repl} \equiv \frac{1}{V} \ln \left[\left[\exp W_{sp}(J)\right]\right]$$
$$= \sum_{L \geq 0} (2K)^L \sum_{\Delta \in \mathcal{G}_{0,0}(L)} w(\Delta) \qquad (52)$$

according to the discussion of Section 2. We recall that $\mathcal{G}_{0,0}(L)$ is the set of DLCE vacuum graphs with L bare lines that are connected in the generalized DLCE sense. Comparing (51) and (52), the transition from $n = 1$ to $n = 0$ replicas is achieved by keeping only the subset $\overline{\mathcal{G}}_{0,0}(L) \subseteq \mathcal{G}_{0,0}(L)$ of multiple-line graphs that are connected in the original (LCE) sense.

We emphasize that the restriction of DLCEs to QDL-CEs is not an ad hoc (or intuitively motivated) assumption but a derived consequence of the fact that the logarithm is taken before the integration $\int \mathcal{D}J$. This procedure accounts for *all* graphs that contribute to a given order in K. Thus we do have to truncate the series unless the series can be completely summed up, as it happens in exceptional cases.

Without the need for truncation there is an equality sign in (51). This should be stressed in contrast to other graphical expansion schemes that remain to be approximative even to arbitrarily high orders. This is certainly an advantage from a systematic point of view. From a practical point of view a computer implementation of the algorithmic generation of graphs is necessary, because the number of contributing graphs increases rather rapidly with the order of the expansion, as explicit calculations in the $SU(2)$ Higgs model have shown [14].

We expect that the series (51) are convergent for a large class of interactions $S^1(J)$ and $v(x, y)$ if the coupling constant K is sufficiently small. For special interactions most of the multiple-line graphs yield vanishing contributions so that we can further restrict the sum to a subset of $\overline{\mathcal{G}}_{0,0}(L)$. An example is given by the mean field type of interaction of the Sherrington-Kirkpatrick model, *cf.* Section 3.2.

3.2 QDLCEs

In the following we list some examples for systems of which we can study the phase structure by means of QDLCEs.

Their actions are special cases of (33–35) with $n = 1$ (as explained above, $n = 1$ does not imply complete annealing here!) and the following choice of variables.

- Infinite range models. Choose $J_{(i,j)} \in \mathbb{R}$ as before, $\sigma_i \in \pm 1$, $i, j \in \{1, \ldots, N\}$,

$$v(x, y) = K(1 - \delta_{x,y}),$$
$$\overline{\Lambda}_1 \equiv \Lambda_1 \text{ is the set of all pairs of sites,}$$
$$S^1(J) = N \frac{1}{2} J^2. \tag{53}$$

Now the sum over the sites in (33) runs over arbitrary pairs (i, j), $i < j$, and we obtain the infinite range Sherrington-Kirkpatrick model. For infinite range and in the thermodynamic limit $(N \rightarrow \infty)$, the phase structure can be solved by replica mean field theory, *cf. e.g.* [16]. For QDLCEs the infinite range and $N \rightarrow \infty$ limits imply that only tree graphs of 2-lines contribute to the series of the free energy density, such as

$$\tag{54}$$
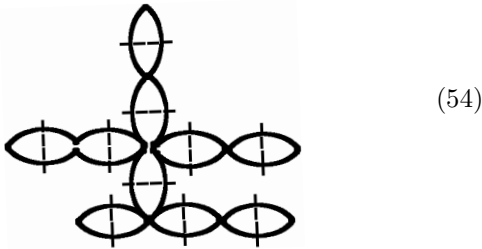
The reason for that is that each 2-line gets a factor of $1/N$ from the S_1-part of the action, but each vertex gets a factor $(N - 1)$ from the embedding onto a lattice Λ_0. The contribution of every tree graph to the free energy is proportional to N. If a chain of 2-lines connecting the vertices gets closed, forming a loop, there is one $(N - 1)$ less in the total embedding factor. Thus the contribution is suppressed by $1/(N - 1)$ for every loop and vanishes in the thermodynamic limit. Because of the simple tree structure there is a chance for summing up the series. This is currently under investigation.

- Finite range connectivity. The sum $\sum_{i<j}$ of the spins is now restricted to next-neighbours or, more generally, to a finite number of pairs. Rather than specifying $S_1(J_{(i,j)})$ of (34), it is sufficient for DLCEs and QDLCEs to choose $\exp(-S_1(J_{(i,j)}))$. Let

$$\exp\left(-S_1(J)\right) = (1-p)\delta(J) + p\delta(J-1) \tag{55}$$

with $p \in [0, 1]$. The variables $J_{(i,j)} \in \{0, 1\}$ can then be interpreted as occupation numbers of the bonds. Furthermore, if we choose $\sigma_i \in \{\pm 1\}$ we obtain a

− bond-diluted Ising model.

Choosing $\sigma_i \in \mathbb{Z}_q$ we obtain a

− bond-diluted q-state Potts model.

If $\sigma_i \in S_q$, we obtain a

− bond-diluted Heisenberg model.

QDLCEs provide a *systematic* analytic expansion for disordered systems with bond dilution in a quenched limit without intrinsic truncation to subsets of graphs to all orders. Coming from the high temperature (small β) region one can study the phase structure as a function of the degree of dilution. While the low orders of the expansion can be easily calculated by hand, the first step towards the phase structure down to the critical region is a computer implementation of the algorithmic generation of graphs. Work in this direction is in progress.

4 Summary and conclusions

In this paper we have explicitly shown how to avoid the replica trick for calculating the free energy of disordered systems in the quenched limit. No uncontrolled limit $n \rightarrow 0$ has to be taken, the quenched system can be represented as a sum of a certain subset of DLCE graphs, instead. This subset of so-called QDLCE graphs has been identified in this paper.

DLCEs are a systematic expansion method to study the phase structure of disordered systems. It is systematic in the sense that we do not restrict the expansion to certain subclasses of graphs that can be summed up, but we identify and keep *all* graphs that contribute to a given order in the expansion parameter. This is certainly an advantage over other graphical expansion schemes. DLCEs provide an analytic tool for studying systems in situations in which it has been impossible so far.

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